Topology of the Set of Smooth Solutions to the Liouville Equation

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Abstract

We prove that the space of smooth initial data and the set of smooth solutions of the Liouville equation are homeomorphic.

1 Introduction

There is a considerable need for the method which can be used to the quantization of *nonlinear* field theories. We still do not know how to quantize, in mathematically correct way, the Einstein theory of gravitation and the Yang-Mills theories. In the case of simple systems with *finitely* many degrees of freedom the method of geometric quantization (see, e.g., [5]) seems to be satisfactory. It is not clear, however, if we can generalize this method to the case of nonlinear field theories. The main problem is that such theories have *infinitely* many degrees of freedom and the set of solutions to the field equations *cannot be* a vector space. The 2-dim Liouville field equation [2]

$$\left(\partial_t^2 - \partial_x^2\right) F(t,x) + \frac{m^2}{2} \exp F(t,x) = 0, \qquad m > 0$$

is a simple model that we are going to use for verification of ideas connected with generalization of the geometric quantization method to nonlinear field theories.

In the geometric quantization procedure one assumes that the phase space of a given classical theory is a *manifold*. In the case of simple systems with n-degrees of freedom one can easily identify the phase space to be a manifold modelled on $\mathbf{R}^{2n}$. In the case of field theory we would like to have an object which we could call a manifold modelled on a Fréchet space. The aim of this paper is to show that the space of smooth initial data and the set of smooth solutions to the Liouville equation are *homeomorphic*. Therefore, the set of solutions has the structure of a topological manifold modelled on the space of initial data.

It follows from the existence of homeomorphism that if there exists a series of smooth initial data convergent almost uniformly with all its derivatives to $(f,g) \in C^\infty(\mathbf{R}) \times C^\infty(\mathbf{R})$, then the series of corresponding solutions of the Liouville equation converges almost uniformly with all its derivatives to the solution corresponding to the initial data (f,g).

Notation : Through the paper we use the spaces $C^\infty(\mathbf{R}^M, \mathbf{R}^N)$,
where $M, N \in \mathbf{N}^\times$; $\mathbf{N}^\times := \mathbf{N} \setminus \{0\}$, $\mathbf{N} = \{0, 1 \ldots\}$;
for $\alpha \in \mathbf{N}^\times$, $K_\alpha := [\,-\alpha, \alpha\,]$, thus $\mathbf{R}^M = \bigcup_{\alpha=1}^\infty K_\alpha^M$.
If $\beta = (\beta_1, \ldots, \beta_M) \in \mathbf{N}^M$, then $\partial^\beta := \partial_1^{\beta_1} \cdot \partial_2^{\beta_2} \cdot \ldots \cdot \partial_M^{\beta_M}$, $\mid \beta \mid := \sum_{j=1}^M \beta_j$.

Topology on $C^\infty(\mathbf{R}^M, \mathbf{R}^N)$ is given by the family of seminorms $\left(p^{MN}_{\alpha\beta}\right)_{(\alpha,\beta)\in\mathbf{N}^\times\times\mathbf{N}^M}$ defined by

$$p^{MN}_{\alpha\beta} : C^\infty(\mathbf{R}^M, \mathbf{R}^N) \ni f \longrightarrow p^{MN}_{\alpha\beta}(f) := \sup_{x\in K^M_\alpha} \| (\partial^\beta f)(x) \| \in \mathbf{R},$$

where

$$\| \cdot \| : \mathbf{R}^N \ni y \to \| y \| := \left(\sum_{j=1}^N (y^j)^2 \right)^{\frac{1}{2}} \in \mathbf{R}.$$

Chapter III of [4] is our source of information on topological vector spaces (in particular Fréchet spaces).
To denote maps $\mathbf{R} \to \mathbf{R}$ we use small Latin letters, $(f, g, h, \ldots \in C^\infty(\mathbf{R}) := C^\infty(\mathbf{R}, \mathbf{R}))$; for maps $\mathbf{R}^2 \to \mathbf{R}$ capital Latin letters are used, $(F, G, H, \ldots \in C^\infty(\mathbf{R}^2))$; $\mathbf{R} \to \mathbf{R}^2$ maps are denoted by capital Greek letters, $(\Phi, \Psi, \Omega, \ldots \in C^\infty(\mathbf{R}, \mathbf{R}^2))$. To shorten notation, we write $p_{\alpha\beta} := p^{11}_{\alpha\beta}$, $r_{\alpha\beta} := p^{12}_{\alpha\beta}$, $q_{\alpha\beta} := p^{21}_{\alpha\beta}$ for $\beta = (\beta_1, \beta_2)$; $\partial_t = \partial_1$, $\partial_x = \partial_2$ for $(t, x) \in \mathbf{R}^2$. A norm of a linear map $A \in B(\mathbf{R}^2) := B(\mathbf{R}^2, \mathbf{R}^2)$ is defined by

$$\| \cdot \| : B(\mathbf{R}^2) \ni A \to \| A \| := \sup_{\|x\|=1} \| A(x) \| \in \mathbf{R}.$$

$C^\infty_+(\mathbf{R}^2) := \{F \in C^\infty(\mathbf{R}^2) : F(\mathbf{R}^2) \subseteq \,]\, 0, \infty\, [\, \}$ is a topological space with topology induced from $C^\infty(\mathbf{R}^2)$. From now on, $\Box := \partial^2_t - \partial^2_x$,
$\mathcal{M} := \{F \in C^\infty(\mathbf{R}^2) : \Box F = -(m^2/2) \exp F\}$, where $m > 0$ is a fixed real number.

In Sec.(2) we quote some results of [1] concerning the solution of the Cauchy problem for the Liouville equation and we prove these results. The proof that the space of smooth initial data and the set of smooth solutions are homeomorphic is given in Sec.(3). We make some remarks in the last Section.

2 Smooth Solutions of the Liouville Equation

We examine properties of a smooth solution to the Liouville equation, i.e., of class $C^\infty(\mathcal{O})$, where $\mathbf{R}^2 \supseteq \mathcal{O}$ is an open subset different to $\emptyset$. However, all proofs can be easily modified to include the solutions of class $C^k(\mathbf{R}^2)$, for $k \geq 2$.

Lemma 1 *Let* $g_i \in C^\infty(\mathbf{R})$ *for* $i \in \overline{1,4}$ *are such that*

$$g_1 g_3' - g_1' g_3 = 1, \quad g_2 g_4' - g_2' g_4 = -1 \tag{1}$$

and let

$$G : \mathbf{R}^2 \ni (t,x) \to G(t,x) := g_1(x+t)g_2(x-t) + g_3(x+t)g_4(x-t) \in \mathbf{R}.$$

Then we have

$$\left(G^{-1}(0) \neq \emptyset\right) \Longrightarrow \left(G^{-1}(0) \cap \{(0,x) \in \mathbf{R}^2 : x \in \mathbf{R}\} \neq \emptyset\right).$$

Proof. Let $\xi := x+t, \quad \eta := x-t$. Making use of

$$(G(t_0,x_0) = 0) \Longleftrightarrow (g_1(\xi_0)g_2(\eta_0) = -g_3(\xi_0)g_4(\eta_0)) \tag{2}$$

we see that the condition

$$\begin{cases} G(t_0,x_0) = 0 \\ (\partial_t G)(t_0,x_0) = (\partial_x G)(t_0,x_0) \end{cases}$$

leads to

$$g_1(\xi_0)g_2'(\eta_0) + g_3(\xi_0)g_4'(\eta_0) = 0.$$

Multiplying this equation by $g_2(\eta_0)$ and using (1) gives $g_3(\xi_0) = 0$. Similarly, multiplying by $g_4(\eta_0)$ leads to $g_1(\xi_0) = 0$. Thus, we have

$$\left(\begin{array}{ccc} G(t_0,x_0) & = & 0 \\ (\partial_t G)(t_0,x_0) & = & (\partial_x G)(t_0,x_0) \end{array}\right) \Longrightarrow (g_1(t_0,x_0) = 0 = g_3(t_0,x_0)),$$

contrary to (1). In the same manner we can see that

$$\left(\begin{array}{ccc} G(t_0,x_0) & = & 0 \\ (\partial_t G)(t_0,x_0) & = & -(\partial_x G)(t_0,x_0) \end{array}\right) \Longrightarrow (g_2(t_0,x_0) = 0 = g_4(t_0,x_0)),$$

which again contradicts (1). Therefore, we have

$$\left((t_0,x_0) \in G^{-1}(0)\right) \Longrightarrow ((\partial_t G)(t_0,x_0) \neq \pm(\partial_x G)(t_0,x_0)). \tag{3}$$

The condition $(\partial_x G)(t_0,x_0) = 0$ means that

$$0 = g_1'(\xi_0)g_2(\eta_0) + g_1(\xi_0)g_2'(\eta_0) + g_3'(\xi_0)g_4(\eta_0) + g_3(\xi_0)g_4'(\eta_0). \tag{4}$$

Multiplying (4) by $g_1(\xi_0)g_4(\eta_0)$ and using (2) gives

$$g_1(\xi_0)^2 + g_4(\eta_0)^2 = 0. \tag{5}$$

Similarly, multiplying (4) by $g_3(\xi_0)g_2(\eta_0)$ and using (2) leads to

$$g_3(\xi_0)^2 + g_2(\eta_0)^2 = 0. \tag{6}$$

Since (5) and (6) contradict (1), we conclude that

$$\big((t_0, x_0) \in G^{-1}(0)\big) \Longrightarrow ((\partial_x G)(t_0, x_0) \neq 0)\,, \tag{7}$$

which means that either $G^{-1}(0) = \emptyset$ or $G^{-1}(0)$ is a one-dimensional C^∞ submanifold of $\mathbf{R}^2$. Suppose that $G^{-1}(0) \neq \emptyset$ and let us denote by M the connected component of $G^{-1}(0)$. By (3) we have that M cannot be a compact subset of $\mathbf{R}^2$. Since M is closed in $\mathbf{R}^2$, it cannot be bounded in $\mathbf{R}^2$. From (7) we conclude that M can be parametrized by $t \in \mathbf{R}$. Let $\pi : \mathbf{R}^2 \ni (t, x) \to \pi(t, x) := t \in \mathbf{R}$. Since M is closed, $\pi(M) \subseteq \mathbf{R}$ is closed in $\mathbf{R}$. Hence $\emptyset \neq \pi(M) \subseteq \mathbf{R}$ is both closed and open (homeomorphic to $\mathbf{R}$). Therefore $\pi(M) = \mathbf{R}$. Finally, we obtain

$$G^{-1}(0) \cap \{(0, x) \in \mathbf{R}^2 : x \in \mathbf{R}\} \neq \emptyset. \qquad \Box$$

By [1] we get

Lemma 2 *Let $\mathbf{R}^2 \supseteq \mathcal{O}$ be an open subset and let*

$$\mathcal{O}_1 := \{(x+t) \in \mathbf{R} \ : \ (t, x) \in \mathcal{O}\}, \quad \mathcal{O}_2 := \{(x-t) \in \mathbf{R} \ : \ (t, x) \in \mathcal{O}\}.$$

Suppose $F \in C^\infty(\mathcal{O})$, then the following are equivalent:

1. $\Box F = -\frac{m^2}{2} \exp F$.

2. *There exist $g_1, g_3 \in C^\infty(\mathcal{O}_1)$ and $g_2, g_4 \in C^\infty(\mathcal{O}_2)$ satisfying $g_1 g_3' - g_1' g_3 = 1$ and $g_2 g_4' - g_2' g_4 = -1$ such that $F : \mathcal{O} \ni (t, x) \to F(t, x) := -\log \frac{m^2}{16} \left[g_1(x+t)g_2(x-t) + g_3(x+t)g_4(x-t)\right]^2 \in \mathbf{R}$.*

Lemmas (1) and (2) lead to

Corollary 1 *If $\mathbf{R}^2 \supseteq \mathcal{O}$ is an open set, such that $\{(0, x) \in \mathbf{R}^2 \ : \ x \in \mathbf{R}\} \subseteq \mathcal{O}$ and $F \in C^\infty(\mathcal{O})$ satisfies the Liouville equation $\Box F = -(m^2/2) \exp F$, then there exists $\widetilde{F} \in C^\infty(\mathbf{R}^2)$ such that $\Box \widetilde{F} = -(m^2/2) \exp \widetilde{F}$ and $\widetilde{F}_{|\mathcal{O}} = F$.*

Proposition 1 *Suppose* $f_1, f_2, g_1, g_2, g_3, g_4 \in C^\infty(\mathbf{R})$ *and let*

$$g_1 g_3' - g_1' g_3 = 1, \tag{8}$$

$$g_2 g_4' - g_2' g_4 = -1, \tag{9}$$

$$\begin{aligned} G : \mathbf{R}^2 \ni (t,x) \to G(t,x) \quad := \quad & g_1(x+t) g_2(x-t) + \\ & g_3(x+t) g_4(x-t) \in \mathbf{R}, \end{aligned} \tag{10}$$

$$u := \frac{1}{16}\left[(f_1' - f_2)^2 - 4(f_1' - f_2)' + m^2 \exp f_1\right], \tag{11}$$

$$w := \frac{1}{16}\left[(f_1' + f_2)^2 - 4(f_1' + f_2)' + m^2 \exp f_1\right]. \tag{12}$$

Then

1. *If* $g_1, \dots, g_4$ *satisfy the equations*

$$g_i'' = u g_i \quad \text{for} \quad i = 2, 4 \tag{13}$$

and

$$g_j'' = w g_j \quad \text{for} \quad j = 1, 3 \tag{14}$$

then the map

$$F : \mathbf{R}^2 \ni (t,x) \to F(t,x) := -\log \frac{m^2}{16} G^2(t,x) \in \mathbf{R} \tag{15}$$

is a solution of the Liouville equation with the initial data

$$\left\{ \begin{array}{rcl} F(0,\cdot) & = & f_1 \\ (\partial_t F)(0,\cdot) & = & f_2. \end{array} \right. \tag{16}$$

2. *A solution of the Liouville equation satisfying (16) is given by (15), where* $g_1, \dots, g_4$ *satisfy (13) and (14).*

Proof (1.) Suppose g_1, g_3 and $\widetilde{g}_1, \widetilde{g}_3$ satisfy (8) and (14). Hence there exists $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{R})$ such that $g_1 = a\widetilde{g}_1 + b\widetilde{g}_3$ and $g_3 = c\widetilde{g}_1 + d\widetilde{g}_3$. Such an exchange of functions corresponds to the Bianchi transformation [1] and does not change the form of solution (15). If g_2 and g_4 satisfy (9) and (13), then the functions [1]

$$g_1 := -\frac{4}{m}\exp\left(-\frac{1}{2}f_1(\cdot)\right)\left[g_4' + \frac{1}{4}(f_1' - f_2)g_4\right], \tag{17}$$

$$g_3 := \frac{4}{m}\exp\left(-\frac{1}{2}f_1(\cdot)\right)\left[g_2' + \frac{1}{4}(f_1' - f_2)g_2\right] \tag{18}$$

satisfy (8) and (14). In what follows we assume that the general form of g_1 and g_3 are given by (17) and (18). One can easily check that for G defined by (10) we have

$$\forall x \in \mathbf{R} \ : \ G(0, x) = \frac{4}{m}\exp\left(-\frac{1}{2}f_1(\cdot)\right) > 0.$$

By Lemma 1 we have that F given by (15) is well defined on $\mathbf{R}^2$ ($F \in C^\infty(\mathbf{R}^2)$) and satisfies (16).
(2.) We have shown that there exists $F \in C^\infty(\mathbf{R}^2)$ satisfying (16). By Lemma 2, F is of the form (15) for $g_1, \ldots, g_4 \in C^\infty(\mathbf{R})$ satisfying (8) and (9). Now, we shall show (see [1]) that $g_1, \ldots, g_4$ can be a solution of (13) and (14) with u and w given by (11) and (12).
Let

$$\aleph : \mathbf{R} \ni x \to \aleph(x) := G(0, x) \in \mathbf{R}$$

$$\hbar : \mathbf{R} \ni x \to \hbar(x) := g_1'(x)g_2'(x) + g_3'(x)g_4'(x) \in \mathbf{R}.$$

Eq.(16) means that

$$\forall x \in \mathbf{R} \quad \aleph(x) \neq 0$$

and

$$\aleph = \epsilon\frac{4}{m}\exp\left(-\frac{1}{2}f_1(\cdot)\right), \tag{19}$$

where

$$\epsilon := \begin{cases} 1 & \text{for } \aleph > 0 \\ -1 & \text{for } \aleph < 0 \end{cases} .$$

By (19) we get $\aleph\,'/\aleph = -\frac{1}{2}f_1'$ and

$$\begin{aligned} f_2 &= (\partial_t F)(0,\cdot) = \frac{2}{\aleph}\left(\aleph\,' - 2(g_1'g_2 + g_3'g_4)\right) = -f_1' - \frac{4}{\aleph}(g_1'g_2 + g_3'g_4) \\ &= \frac{2}{\aleph}\left(2(g_1g_2' + g_3'g_4) - \aleph\,'\right) = f_1' + \frac{4}{\aleph}(g_1g_2' + g_3g_4') \end{aligned}$$

which leads to

$$\frac{1}{4}(f_1' + f_2)\aleph = -g_1'g_2 - g_3'g_4, \tag{20}$$

$$\frac{1}{4}(f_1' - f_2)\aleph = g_1g_2' + g_3g_4'. \tag{21}$$

Suppose now that g_1, g_3 satisfy (13) and g_2, g_4 satisfy (14) for some $u, w \in C^\infty(\mathbf{R})$. Taking derivative of (20) yields

$$\frac{1}{4}(f_2 + f_1')'\,\aleph + \frac{1}{4}(f_2 + f_1')\,\aleph\,' = -w\aleph - \hbar \tag{22}$$

By analogy, we get from (21):

$$\frac{1}{4}(f_2 - f_1')'\,\aleph + \frac{1}{4}(f_2 - f_1')\,\aleph\,' = u\,\aleph + \hbar \tag{23}$$

By direct calculations, we get

$$(\partial_t^2 F)(0,\cdot) = \frac{1}{2}f_2^2 + 4\frac{\hbar}{\aleph} - 2(u + w) \tag{24}$$

Since $(\partial_x^2 F)(0,\cdot) = f_1''$, we get

$$f_1'' = 2\left(\frac{\aleph\,'}{\aleph}\right)^2 - 2\frac{\aleph\,''}{\aleph} \tag{25}$$

The map F satisfies the Liouville equation $\Box F = -(m^2/2)\exp F$ on $\mathbf{R}^2$. By (24) and (25), for $(0,x) \in \mathbf{R}^2$ we get

$$\frac{1}{2}f_2^2 + 4\frac{\hbar}{\aleph} - 2(u + w) + 2\frac{\aleph\,''}{\aleph} - 2\left(\frac{\aleph\,'}{\aleph}\right)^2 = -\frac{m^2}{2}\exp f_1. \tag{26}$$

Since

$$2\,\aleph''/\aleph = 2(u+w) + 4\,\hbar/\aleph \quad \text{and} \quad (\aleph'/\aleph)^2 = (1/4)(f_1')^2$$

Eq. (26) leads to

$$\frac{\hbar}{\aleph} = -\frac{1}{16}\left[(f_2 - f_1')(f_2 + f_1') - m^2 \exp f_1\right]. \tag{27}$$

By (27) and (22) we get

$$w = \frac{1}{16}(f_1' - f_2)^2 - \frac{1}{4}(f_1' - f_2)' + \frac{m^2}{16}\exp f_1. \tag{28}$$

Similarly, (27) and (23) give

$$u = \frac{1}{16}(f_1' + f_2)^2 - \frac{1}{4}(f_1' + f_2)' + \frac{m^2}{16}\exp f_1. \quad \square \tag{29}$$

3 Homeomorphism of the Space of Initial Data and the Set of Solutions

We define the following mappings:

$$\mathcal{A} : C^\infty(\mathbf{R})^2 \ni (f_1, f_2) \longrightarrow \mathcal{A}(f_1, f_2) := u \in C^\infty(\mathbf{R}), \tag{30}$$

where u is given by (11).

$$\mathcal{B} : C^\infty(\mathbf{R}) \ni u \longrightarrow \mathcal{B}(u) := (g_2, g_4) \in C^\infty(\mathbf{R})^2, \tag{31}$$

where g_2 and g_4 are defined by

$$\begin{cases} g_2'' &= u g_2 \\ g_2(0) &= 0 \\ g_2'(0) &= 1 \end{cases} \quad \text{and} \quad \begin{cases} g_4'' &= u g_4 \\ g_4(0) &= 1 \\ g_4'(0) &= 0 \end{cases}$$

Remark 1 *Maps g_2 and g_4 satisfy (9).*

$$\mathcal{C} : C^\infty(\mathbf{R})^4 \ni (f_1, f_2, g_2, g_4) \longrightarrow \mathcal{C}(f_1, f_2, g_2, g_4) := (g_1, g_3) \in C^\infty(\mathbf{R})^2, \quad (32)$$

where g_1 and g_3 are given by (17) and (18).

$$\mathcal{D} : C^\infty(\mathbf{R})^2 \ni (f_1, f_2) \longrightarrow \mathcal{D}(f_1, f_2) := (f_1, f_2, f_1, f_2) \in C^\infty(\mathbf{R})^4, \quad (33)$$

$$\mathcal{E} : C^\infty(\mathbf{R}^2) \ni G \longrightarrow \mathcal{E}(G) := \frac{m^2}{16} G^2 \in C^\infty(\mathbf{R}^2), \quad (34)$$

$$\mathcal{N} : C^\infty_+(\mathbf{R}^2) \ni H \longrightarrow \mathcal{N}(H) := -\log H \in C^\infty(\mathbf{R}^2), \quad (35)$$

$$\mathcal{G} : C^\infty(\mathbf{R})^4 \ni (g_1, g_3, g_2, g_4) \longrightarrow \mathcal{G}(g_1, g_3, g_2, g_4) := G \in C^\infty(\mathbf{R}^2), \quad (36)$$

where G is defined by (10).

$$\mathcal{H} := \mathcal{E} \circ \mathcal{G} \circ (\mathcal{C} \times id_2) \circ (id_2 \times \mathcal{D}) \circ (id_2 \times \mathcal{B}) \circ (id_2 \times \mathcal{A}) \circ \mathcal{D}, \quad (37)$$

where $id_2 := id_{C^\infty(\mathbf{R})^2}$.

Remark 2 $\mathcal{H} : C^\infty(\mathbf{R})^2 \ni (f_1, f_2) \longrightarrow \mathcal{H}(f_1, f_2) := \frac{m^2}{2} G^2 \in C^\infty(\mathbf{R}^2)$.

By Lemma 1 we have

Corollary 2 $\mathcal{H}\left(C^\infty(\mathbf{R})^2\right) \subseteq C^\infty_+(\mathbf{R}^2)$

Let

$$\mathcal{I} := \mathcal{N} \circ \mathcal{H}. \quad (38)$$

By Proposition 1 we get

Corollary 3 $\mathcal{I} : C^\infty(\mathbf{R})^2 \longrightarrow \mathcal{M}$ *is a bijection.*

Now comes the main theorem.

Theorem 1 *The mapping* $\mathcal{I} : C^\infty(\mathbf{R})^2 \longrightarrow \mathcal{M}$ *defined by (38) is a homeomorphism.*

Before we give the proof, let us prove a few Lemmas.
We define some auxiliary maps and sets.
For $\beta \in \mathbf{N}^{\times}$:

$$\mathcal{R}(\beta) := \{a \in \mathbf{N}^{\beta} : \sum_{j=1}^{\beta} j a_j = \beta\},$$

$$\mathcal{P}_{\beta} : \mathcal{R}(\beta) \ni a \longrightarrow \mathcal{P}_{\beta}(a) := \frac{\beta!}{\prod_{j=1}^{\beta} (j!)^{a_j} a_j!} \in \mathbf{N},$$

$$l_{\beta} : \mathcal{R}(\beta) \ni a \longrightarrow l_{\beta}(a) := \sum_{j=1}^{\beta} a_j \in \mathbf{N}.$$

For $(\lambda, \mu) \in \mathbf{N}^{\times} \times \mathbf{N}$:

$$\mathcal{R}(\lambda, \mu) := \{a \in \mathbf{N}^{\lambda+1} : \sum_{j=1}^{\lambda} j a_j = \lambda \ , \ \sum_{j=0}^{\lambda} a_j = \mu\},$$

$$\mathcal{W}_{\lambda,\mu} : \mathcal{R}(\lambda, \mu) \ni a \longrightarrow \mathcal{W}_{\lambda,\mu}(a) := \frac{\mu!}{a_0} \frac{\lambda!}{\prod_{j=1}^{\lambda} (j!)^{a_j} a_j!} \in \mathbf{N},$$

$$c := (c^1, \dots, c^{\beta}) \in \mathop{X}_{i=1}^{\beta} \mathcal{R}(\lambda_i, \mu_i),$$

$$\mathcal{T}(\lambda, \mu) := \{a \in \mathbf{N}^{\lambda} : \sum_{j=1}^{\lambda} a_j = \mu\},$$

$$\mathcal{N}_{\lambda,\mu} : \mathcal{T}(\lambda, \mu) \ni a \longrightarrow \mathcal{N}_{\lambda,\mu}(a) := \frac{\mu!}{\prod_{j=1}^{\lambda} a_j!} \in \mathbf{N}.$$

Lemma 3 *Let* $\mathbf{R} \supseteq \mathcal{O}_1$ *and* $\mathbf{R}^2 \supseteq \mathcal{O}_2$ *be some open sets,*

$$h \in C^\infty(\mathcal{O}_1), \quad J \in C^\infty(\mathcal{O}_2) \quad \textit{and} \quad \forall (t,x) \in \mathcal{O}_2 : 1 + J(t,x) > 0.$$

Then

1. $\forall \beta \in \mathbf{N}^\times : \ \partial^\beta \exp h = \left(\sum\limits_{a \in \mathcal{R}(\beta)} \mathcal{P}_\beta(a) \prod\limits_{j=1}^{\beta} (\partial^j h)^{a_j} \right) \exp h.$

2. $\forall \beta \in \mathbf{N}^\times \quad \forall i \in \{1,2\}$:

$$\partial_i^\beta \log(1+J) = - \sum_{a \in \mathcal{R}(\beta)} (-1)^{l_\beta(a)} \frac{(l_\beta(a)-1)!}{(1+J)^{l_\beta(a)}} \, \mathcal{P}_\beta(a) \prod_{j=1}^{\beta} (\partial_i^j J)^{a_j}.$$

3. $\forall \beta, \gamma \in \mathbf{N}^\times : \ \partial_1^\gamma \partial_2^\beta \log(1+J) =$

$$- \sum_{b \in \mathcal{R}(\gamma)} \sum_{a \in \mathcal{R}(\beta)} (-1)^{l_\beta(a) + l_\gamma(b)} \ \frac{(l_\beta(a) + l_\gamma(b) - 1)!}{(1+J)^{l_\beta(a)+l_\gamma(b)}} \mathcal{P}_\beta(a) \mathcal{P}_\gamma(b) \ \times$$

$$\prod_{j=1}^{\beta} \prod_{k=1}^{\gamma} (\partial_1^j J)^{a_j} (\partial_2^k J)^{b_k} - \sum_{a \in \mathcal{R}(\beta)} \ \sum_{b \in \mathcal{T}(\beta,\gamma)} (-1)^{l_\beta(a)} \ \frac{(l_\beta(a)-1)!}{(1+J)^{l_\beta(a)}} \ \times$$

$$\mathcal{P}_\beta(a) \mathcal{N}_{\beta,\gamma}(b) \sum_{c \in \mathop{X}\limits_{i=1}^{\beta} \mathcal{R}(b_i, a_i)} \prod_{k=1}^{\beta} \mathcal{W}_{b_k, a_k}(c^k) \prod_{j=1}^{b_k} (\partial_1^j \partial_2^k J)^{c_j^k}.$$

Remark 3 *One knows that*

$$\forall N \in \mathbf{N} \quad \forall k \in \mathbf{N}^\times : \ | \{ x \in \mathbf{N}^k : \sum_{j=1}^{k} x_j = N \} | = \begin{pmatrix} N+k-1 \\ k-1 \end{pmatrix}.$$

Thus

$$\forall \ \beta \in \mathbf{N}^\times \ : \ | \, \mathbf{R}(\beta) \, | \le \sum_{j=1}^{\beta} \begin{pmatrix} \beta + k - 1 \\ k-1 \end{pmatrix} = \begin{pmatrix} 2\beta \\ \beta - 1 \end{pmatrix} < \infty.$$

Therefore, all sums in Lemma 3 are finite.

We skip a simple but lenghty proof of the Lemma 3.

Lemma 4 *Let* (X, d_X) , (Z, d_Z) *be some metric spaces and let* (Y, p_Y) *be a semimetric space. If* $X \supseteq K$ *is compact and* $Q : K \times Y \to Z$ *is a continuous map, then*

$$\forall\, \epsilon > 0 \quad \forall\, y_0 \in Y \quad \exists\, \delta > 0 :$$

$$(y \in K(y_0, \delta)) \Longrightarrow (\forall\, x \in K : d_Z(Q(x, y), Q(x, y_0)) < \epsilon)\,.$$

Proof of this Lemma results from the proof of Lemma IX.3.1 of [3] .
Proof of Theorem 1:
Step 1. Let $(f_n)_{n=0}^{\infty}$ and $(g_n)_{n=0}^{\infty}$ are two sequences convergent in $C^\infty(\mathbf{R})$ to f and g , correspondingly. Then,

$$\forall (\mu, \nu) \in \mathbf{N}^\times \times \mathbf{N} \quad \exists c_{\mu\nu} \in \mathbf{R} \quad \forall n \in \mathbf{N} : p_{\mu\nu}(f_n) \leq c_{\mu\nu}.$$

Let us fix $c_{\mu\nu}$ for $(\mu, \nu) \in \mathbf{N}^\times \times \mathbf{N}$
(e.g., $c_{\mu\nu} := \inf\{\tilde{c}_{\mu\nu} \in \mathbf{R} : \forall\, n \in \mathbf{N} \;\; p_{\mu\nu}(f_n) \leq \tilde{c}_{\mu\nu}\}$)
and denote

$$M_1 := \max\left\{ \left(\begin{array}{c} \beta \\ \gamma \end{array} \right) c_{\alpha\gamma} \; : \; \gamma \in \{0, 1, \ldots, \beta\} \right\},$$

$$M_2 := \max\left\{ \left(\begin{array}{c} \beta \\ \gamma \end{array} \right) p_{\alpha(\gamma-\beta)}(g) \; : \; \gamma \in \{0, 1, \ldots, \beta\} \right\},$$

$$M_{\alpha\beta} := \max\{M_1, M_2\}.$$

Then,

$$\begin{aligned}
&p_{\alpha\beta}(f_n g_n - fg) \leq \\
&\quad \sum_{\gamma=0}^{\beta} \left(\begin{array}{c} \beta \\ \gamma \end{array} \right) p_{\alpha\gamma}(f_n) p_{\alpha(\beta-\gamma)}(g_n - g) + \sum_{\gamma=0}^{\beta} \left(\begin{array}{c} \beta \\ \gamma \end{array} \right) p_{\alpha\gamma}(f_n - f) p_{\alpha(\beta-\gamma)}(g) \leq \\
&\quad M_{\alpha\beta} \sum_{\gamma=0}^{\beta} \left(p_{\alpha(\beta-\gamma)}(g_n - g) + p_{\alpha\gamma}(f_n - f) \right) \underset{n\to\infty}{\longrightarrow} 0
\end{aligned}$$

which means that the mapping

$$C^\infty(\mathbf{R})^2 \ni (f,g) \to fg \in C^\infty(\mathbf{R})$$

is continuous.

Define $h_n := f_n - f$ for $n \in \mathbf{N}$. We have $e^f - e^{f_n} = e^f(1 - e^{h_n})$, thus

$$p_{\alpha\beta}(e^f - e^{f_n}) \le \sum_{\gamma=0}^{\beta} \left(\begin{array}{c} \beta \\ \gamma \end{array} \right) p_{\alpha\gamma}(e^f) p_{\alpha(\beta-\gamma)}(1 - e^{h_n}).$$

We get

$$p_{\alpha\beta}(e^f - e^{f_n}) \le \tilde{M} \sum_{\gamma=0}^{\beta} p_{\alpha(\beta-\gamma)}(1 - e^{h_n}),$$

where

$$\tilde{M} := \max\left\{ \left(\begin{array}{c} \beta \\ \gamma \end{array} \right) p_{\alpha\gamma}(e^f) : \gamma \in \{0,1,\ldots,\beta\} \right\} \in \mathbf{R}.$$

For $\gamma = \beta$ we have

$$p_{\alpha 0}(1 - e^{h_n}) \le e^{p_{\alpha 0}(h_n)} - 1 \underset{n\to\infty}{\longrightarrow} 0\,.$$

For $\gamma > \beta \ge 0$ denote $\beta - \gamma := \rho + 1$ (so $\rho \ge 0$), then

$$p_{\alpha(\rho+1)}(1 - e^{h_n}) \le \sum_{\mu=0}^{\rho} \left(\begin{array}{c} \rho \\ \mu \end{array} \right) p_{\alpha(\mu+1)}(h_n) p_{\alpha(\rho-\mu)}(e^{h_n}). \tag{39}$$

By Lemma 3 (see *1.*) we have

$$\exists M_0 \in \mathbf{R} \quad \forall \mu \in \overline{0,\rho} \quad \forall\, n \in \mathbf{N} \;:\; \left(\begin{array}{c} \rho \\ \mu \end{array} \right) p_{\alpha(\rho-\mu)}(e^{h_n}) < M_0,$$

therefore (39) gives

$$\forall (\alpha,\beta) \ni \mathbf{N}^\times \times \mathbf{N} : p_{\alpha\beta}(e^f - e^{f_n}) \underset{n\to\infty}{\longrightarrow} 0\,.$$

Since the map

$$\partial : C^\infty(\mathbf{R}) \ni f \longrightarrow \partial f \in C^\infty(\mathbf{R})$$

is continuos, the mapping $\mathcal{A}$ defined by (30) is continuous as the composition of continuous maps.

Step 2. Suppose $u \in C^\infty(\mathbf{R})$ and $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbf{R}^2$. There is one and only one function $g \in C^\infty(\mathbf{R})$ such that

$$\left\{ \begin{array}{lcl} g'' & = & ug \\ g(0) & = & a \\ g'(0) & = & b \end{array} \right. . \tag{40}$$

Making substitution

$$\Psi : \mathbf{R} \ni s \longrightarrow \Psi(s) := \begin{pmatrix} g(s) \\ g'(s) \end{pmatrix} \in \mathbf{R}^2 \tag{41}$$

in (40) yields

$$\dot{\Psi} = \begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix} \Psi, \qquad \Psi(0) = \begin{pmatrix} a \\ b \end{pmatrix}. \tag{42}$$

Let us denote the solution of (42) by $\Psi(\,\cdot\,; u)$, to indicate its dependence on $u \in C^\infty(\mathbf{R})$, and in addition let

$$A : C^\infty(\mathbf{R}) \ni u \longrightarrow A(u) := \begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix} \in C^\infty(\mathbf{R}, M_{2\times 2}(\mathbf{R})),$$

$$B : C^\infty(\mathbf{R}) \ni h \longrightarrow B(h) := \begin{pmatrix} 0 & 0 \\ h & 0 \end{pmatrix} \in C^\infty(\mathbf{R}, M_{2\times 2}(\mathbf{R})).$$

(In the sequal $A(u)(s) := A(u(s))$, $B(h)(s) := B(h(s))$.)
For $(\alpha, \beta) \in \mathbf{N}^\times \times \mathbf{N}$ and $\delta > 0$ we denote

$$s_{\alpha\beta}(\delta) := \{h \in C^\infty(\mathbf{R}) : \ \forall \gamma \in \overline{0, \beta} \quad p_{\alpha\gamma}(h) < \delta\}.$$

We notice that

$$\forall (u, h) \in C^\infty(\mathbf{R}) \times s_{\alpha 0}(1) : \sup_{t \in K_\alpha} \| A(u+h)(t) \| \leq 1 + p_{\alpha 0}(u),$$

$$\forall\, h \in C^\infty(\mathbf{R}) \ : \ \sup_{t \in K_\alpha} \| B(h)(t) \| = p_{\alpha 0}(h).$$

Now, let us fix $u \in C^{\infty}(\mathbf{R})$ and $\alpha \in \mathbf{N}^{\times}$, and let us denote

$$M_u := 4 + p_{\alpha 0}(u).$$

We choose $\tau \in \left]\ 0, \frac{1}{1+M_u}\ \right[$, $N(\tau) := \min\{n \in \mathbf{N}^{\times} : n\tau \geq \alpha\}$ and $K(\tau) := [-\tau N(\tau), \tau N(\tau)]$. Let $N \in \mathbf{N}^{\times}$ is such that $N \leq \frac{1}{\tau} \leq N+1$ and let $d := \frac{1}{N+2}$. Taking into account that

$$\forall\ t \in \mathbf{R}\ :\ \Psi(t;u) = \left(\begin{array}{c} a \\ b \end{array} \right) + \int_0^t A(u(s))\Psi(s;u)ds$$

we get

$$\forall t \in [0,\tau] : \parallel \Psi(t;u+h) - \Psi(t;u) \parallel \leq$$

$$\mid t \mid M_u \sup_{s\in[0,\tau]} \parallel \Psi(s;u+h) - \Psi(s;u) \parallel + \mid t \mid \sup_{s \in K_\alpha} \parallel B(h(s))\Psi(s;u) \parallel . \quad (43)$$

Let us fix $\epsilon > 0$. Applying Lemma 4 to the map

$$Q_u : \mathbf{R} \times C^{\infty}(\mathbf{R}) \ni (s,h) \longrightarrow Q_u(s,h) := B(h(s))\Psi(s;u) \in \mathbf{R},$$

and $(Y, p_Y) = (C^{\infty}(\mathbf{R}), p_{\alpha 0})$ gives

$$\exists \delta_0 > 0\ \ \forall h \in s_{\alpha 0}(\delta_0) : \sup_{s \in K(\tau)} \parallel B(h(s))\Psi(s;u) \parallel < \epsilon d^{N(\tau)}.$$

Making use of this in (43) yields

$$\forall\ h \in s_{\alpha 0}(\delta_0) : \sup_{t\in[0,\tau]} \parallel \Psi(t;u+h) - \Psi(t;u) \parallel \leq$$
$$\tau M_u \sup_{s\in[0,\tau]} \parallel \Psi(s;u+h) - \Psi(s;u) \parallel + \epsilon d^{N(\tau)}.$$

Since τ is such that $0 < \tau < 1 - \tau M_u$ we get

$$\forall\ h \in s_{\alpha 0}(\delta_0) : \sup_{t\in[0,\tau]} \parallel \Psi(t;u+h) - \Psi(t;u) \parallel \leq \epsilon d^{N(\tau)}.$$

Applying previous considerations to the case $t \in [\tau, 2\tau]$ and making use of

$$\forall\ h \in s_{\alpha 0}(\delta_0) : \parallel \Psi(\tau;u+h) - \Psi(\tau;u) \parallel \leq \epsilon d^{N(\tau)}$$

one gets

$$\forall\ h \in s_{\alpha 0}(\delta_0) : \sup_{s\in[\tau,2\tau]} \parallel \Psi(s;u+h) - \Psi(s;u) \parallel \leq \epsilon d^{N(\tau)-1}.$$

Repeated application of this procedure to $[2\tau, 3\tau], \ldots, (N(\tau)-1)\tau, N(\tau)\tau]$ gives

$$\begin{gathered}\forall h \in s_{\alpha 0}(\delta_0)\ \ \forall N \in \overline{1, N(\tau)}\ : \\ \sup_{s\in[(N-1)\tau,N\tau]} \parallel \Psi(s;u+h) - \Psi(s;u) \parallel \leq \epsilon d^{N(\tau)-(N-1)}.\end{gathered}$$

Similar reasoning applied to $[-\tau, 0], [-2\tau, -\tau], \ldots, [-N(\tau)\tau, -(N(\tau)-1)\tau]$ enables to write

$$\forall\ h \in s_{\alpha 0}(\delta_0) : \sup_{s\in K(\tau)} \parallel \Psi(s;u+h) - \Psi(s;u) \parallel \leq \epsilon.$$

Since $K_\alpha \subseteq K(\tau)$, we get

$$\forall \epsilon > 0\ \ \exists \delta_0 > 0\ \ \forall h \in s_{\alpha 0}(\delta_0) : r_{\alpha 0}(\Psi(\cdot\ ; u+h) - \Psi(\cdot\ ; u)) \leq \epsilon. \tag{44}$$

Making use of Lemma 4, Eq. (44) and the estimate

$$\begin{gathered}\forall\ h \in s_{\alpha 0}(1) : \parallel (\partial\Psi)(t;u+h) - (\partial\Psi)(t;u) \parallel \leq \\ M_u \parallel \Psi(t;u+h) - \Psi(t;u) \parallel + \parallel B(h(t))\Psi(t;u) \parallel\end{gathered}$$

we get

$$\forall \epsilon > 0\ \ \exists \delta_1 > 0\ \ \forall h \in s_{\alpha 1}(\delta_1) : r_{\alpha 1}(\Psi(\cdot\ ; u+h) - \Psi(\cdot\ ; u)) \leq \epsilon.$$

Now, let us consider the identity

$$\forall \beta \in \mathbf{N}^\times : \partial^\beta \left(\Psi(\cdot\ ; u+h) - \Psi(\cdot\ ; u)\right) =$$

$$\sum_{\rho=0}^{\beta-1} \left(\begin{array}{c} \beta-1 \\ \rho \end{array} \right) \left[A(\partial^\rho(u+h))\partial^{\beta-1-\rho}(\Psi(\cdot\ ; u+h) - \Psi(\cdot\ ; u))\right] -$$

$$\sum_{\rho=0}^{\beta-1} \left(\begin{array}{c} \beta-1 \\ \rho \end{array} \right) B(\partial^\rho h)\partial^{\beta-1-\rho}\Psi(\cdot\ ; u). \tag{45}$$

We notice that there are derivatives of order $0, 1, \dots, \beta - 1$ in the right hand side of (45). Using Lemma 4 for the map

$$Q_u^\rho : \mathbf{R} \times C^\infty(\mathbf{R}) \ni (s, h) \longrightarrow Q_u^\rho(s, h) := B(\partial^\rho h)\partial^{\beta-1-\rho}\Psi(s; u) \in \mathbf{R}^2,$$

where $\rho = 0, 1, \dots, \beta - 1$ and where $(Y, p_Y) = (C^\infty(\mathbf{R}), p_{\alpha\rho})$,
we get (by induction)

$$\forall \beta \in \mathbf{N} \;\; \forall \epsilon > 0 \;\; \exists \delta_\beta > 0 \;\; \forall h \in s_{\alpha\beta}(\delta_\beta) : r_{\alpha\beta}(\Psi(\,\cdot\,; u + h) - \Psi(\,\cdot\,; u)) \le \epsilon.$$

Since our considerations apply to any $\alpha \in \mathbf{N}^\times$ and any $u \in C^\infty(\mathbf{R})$ we obtain that

$$C^\infty(\mathbf{R}) \ni u \longrightarrow \Psi(\,\cdot\,; u) \in C^\infty(\mathbf{R}, \mathbf{R}^2) \tag{46}$$

(where $\Psi(\,\cdot\,; u)$ is a solution of (42))
is a continuous mapping. Taking into account that

$$\forall \Phi \in C^\infty(\mathbf{R}, \mathbf{R}^2) \quad \forall (\alpha, \beta) \in \mathbf{N}^\times \times \mathbf{N} : r_{\alpha\beta}(\Phi) \ge p_{\alpha\beta}(\Phi_1),$$

(where $\Phi(t) =: \begin{pmatrix} \Phi_1(t) \\ \Phi_2(t) \end{pmatrix}$) and solving (46) for $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ gives the conclusion that $\mathcal{B}$, defined by (31), is a continuous mapping.

Step 3. It is clear that both mappings $\mathcal{C}$ and $\mathcal{D}$ are continuous. The continuity of $\mathcal{E}$ can be proved by analogy to the case of $\mathcal{A}$ mapping. The continuity of $\mathcal{G}$ mapping results from the continuity of the mapping

$$\forall c \in \{-1, 1\} \quad \omega_c : C^\infty(\mathbf{R}) \ni f \longrightarrow \omega_c(f) \in C^\infty(\mathbf{R}^2),$$

where

$$\omega_c(f) : \mathbf{R}^2 \ni (t, x) \longrightarrow \omega_c(f)(t, x) := f(x + ct) \in \mathbf{R}.$$

The mapping $\mathcal{H}$ is continuous since it is a composition of continuous mappings.

Step 4. What is left is to prove that the mapping $\mathcal{N}$ is continuous. Suppose $(G_n)_{n=0}^\infty$ is a sequence of elements of $C_+^\infty(\mathbf{R}^2)$ convergent to $G \in C_+^\infty(\mathbf{R}^2)$.

Denote $H_n := G_n - G$. The sequence $(H_n)_{n=0}^{\infty}$ converges to zero in $C^{\infty}(\mathbf{R}^2)$. Since $\forall n \in \mathbf{N} : 1 + \frac{H_n}{G} > 0$, we have

$$\log G_n - \log G = \log \frac{G_n}{G} = \log(1 + \frac{H_n}{G}).$$

As $H_n \underset{n\to\infty}{\longrightarrow} 0$ in $C^{\infty}(\mathbf{R}^2)$ we have

$$\forall \alpha \in \mathbf{N}^{\times} \quad \exists N_\alpha \in \mathbf{R} \quad \forall n > N_\alpha : q_{\alpha(0,0)}\left(\frac{H_n}{G}\right) < \frac{1}{2}.$$

Since $\forall x \in]-1/2, \infty[\ :\ \mid \log(1+x) \mid \leq 2 \mid x \mid$, we conclude that $\forall \alpha \in \mathbf{N}^{\times} \quad \exists N_\alpha \in \mathbf{N} \quad \forall n > N_\alpha :$

$$q_{\alpha(0,0)}\left(\log(1 + \frac{H_n}{G})\right) \leq 2 q_{\alpha(0,0)}(\frac{1}{G}) q_{\alpha(0,0)}(H_n).$$

Now, suppose $\mid \beta \mid = \beta_1 + \beta_2 > 0$ and denote $J_n := \frac{H_n}{G}$ for $n \in \mathbf{N}$. Since

$$\partial^{(\beta_1 \beta_2)} J_n = \sum_{\gamma_1=0}^{\beta_1} \sum_{\gamma_2=0}^{\beta_2} \begin{pmatrix} \beta_1 \\ \gamma_1 \end{pmatrix} \begin{pmatrix} \beta_2 \\ \gamma_2 \end{pmatrix} \left(\partial^{(\gamma_1 \gamma_2)} \frac{1}{G}\right) \left(\partial^{(\beta_1 - \gamma_1, \beta_2 - \gamma_2)} H_n\right)$$

we have

$$q_{\alpha(\beta_1,\beta_2)}(J_n) \leq M_{\alpha(\beta_1,\beta_2)} \sum_{\gamma_1=0}^{\beta_1} \sum_{\gamma_2=0}^{\beta_2} q_{\alpha(\gamma_1,\gamma_2)}(H_n),$$

where

$$M_{\alpha(\beta_1,\beta_2)} := \max\left\{ \begin{pmatrix} \beta_1 \\ \gamma_1 \end{pmatrix} \begin{pmatrix} \beta_2 \\ \gamma_2 \end{pmatrix} q_{\alpha(\gamma_1,\gamma_2)}(\frac{1}{G}) : 0 \leq \gamma_1 \leq \beta_1, 0 \leq \gamma_2 \leq \beta_2 \right\}.$$

Thus, we see that $J_n \underset{n\to\infty}{\longrightarrow} 0$ in $C^{\infty}(\mathbf{R}^2)$. In particular we have

$$\forall \alpha \in \mathbf{N}^{\times} \quad \exists N^\alpha \in \mathbf{N} \quad \forall n > N^\alpha : \sup_{(t,x) \in K_\alpha^2} \mid \frac{1}{1 + J_n(t,x)} \mid \leq 2. \tag{47}$$

For $\beta \in \mathbf{N}^2$ let $D_\beta := \{(i,j) \in \mathbf{N}^2 : 1 \leq i + j \leq \mid \beta \mid\}$, $d_\beta := \mid D_\beta \mid$. By (47) and Lemma 3 (see *2.* and *3.*) we get that for $\beta \in \mathbf{N}^2$, $\mid \beta \mid \geq 1$

and $\alpha \in \mathbf{N}^{\times}$ there exists a polynomial $Q_{\alpha\beta} \in \mathbf{R}[\, x_1, \dots, x_{d_\beta} \,]$ such that $Q_{\alpha\beta}(0) = 0, \quad degQ_{\alpha\beta} \leq \mid \beta \mid$ and

$$\exists N^\alpha \in \mathbf{N} \quad \forall n > N^\alpha \quad q_{\alpha(\beta_1,\beta 2)}(\log(1 + J_n)) \leq$$

$$Q_{\alpha\beta}\left(q_{\alpha(1,0)}(J_n), q_{\alpha(0,1)}(J_n), \dots, q_{\alpha(|\beta|,0)}(J_n), q_{\alpha(0,|\beta|)}(J_n)\right).$$

Since $J_n \to 0$, it follows that

$$\forall(\alpha, \beta) \in \mathbf{N}^{\times} \times \mathbf{N}^2 \quad \forall\epsilon > 0 \;\; \exists N_{\alpha\beta} \in \mathbf{N} \;\; \forall n > N_{\alpha\beta} : q_{\alpha\beta}(\log(1 + J_n)) < \epsilon.$$

But $G \in C^\infty_+(\mathbf{R}^2)$ is an arbitrary function, therefore the mapping $\mathcal{N}$ is continuous. Finally, the mapping $\mathcal{I}$, defined by (38), is continuous as it is a composition of continuous mappings.
Step 5. Denote $\mathcal{F} := C^\infty(\mathbf{R})^2$. It is clear that the mappings

$$\mathcal{N} : C^\infty(\mathbf{R}^2) \ni F \longrightarrow \mathcal{N}(F) := (F(0, \cdot), (\partial_t F)(0, \cdot)) \in \mathcal{F}$$

and

$$\mathcal{S} := \mathcal{N}_{|\mathcal{M}}$$

are continuous.
Corrolary 3 means that

$$\mathcal{I} \cdot \mathcal{S} = id_{\mathcal{M}} \quad \text{ and } \quad \mathcal{S} \cdot \mathcal{I} = id_{\mathcal{F}}$$

This completes the proof.

4 Concluding Remarks

Homeomorphism of the set of smooth solutions, $\mathcal{M} \subset C^\infty(\mathbf{R}^2, \mathbf{R})$, and the space of smooth initial data, $\mathcal{F}$, gives $\mathcal{M}$ the structure of a topological manifold modelled on the Fréchet space $\mathcal{F}$.

A starting point in the geometric quantization of a mechanical system is to express the evolution of the system on the phase space in terms of *symplectic* geometry. Can one follow this method in the case of the Liouville field theory? We hope to answer this question in near future.

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